Auditorium Exercise Sheet 4

Differential Equations I for Students of Engineering Sciences

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Bernoulli equation[*]

A first order (non-linear) equation of the form

$$y'(t) = a(t)y(t) + b(t)y(t)^{\alpha}, \quad a, b \in \mathrm{C}(I),\ \alpha \in \mathbb{R} \setminus \{0,1\} \tag{1}$$

$$\text{(necessary } y > 0 \text{ if } \alpha \notin \mathbb{N}\text{)}$$

is called **Bernoulli differential equation**

With the substitution $\boldsymbol{u(t) = y^{1-\alpha}(t)}$, it is $u'(t) = (1-\alpha)y'(t)y(t)^{-\alpha}$, and dividing the equation by y^{α} we get

$$u'(t) = (1-\alpha)\left[a(t)u(t) + b(t)\right] \to \text{first order, linear ODE in } u$$

which can now be solved in u (apply formula or separation of variables). Finally, substitute back $y = u^{1/(1-\alpha)}$.

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[*] From the Swiss mathematician Jacob Bernoulli (1655-1705)

Example 1

Find the general solution of the ODE $y' = y + 2y^5$, for $y = y(t)$.

It is a Bernoulli equation with $a(t) = 1$, $b(t) = 2$ and $\alpha = 5$: we apply the substitution $u(t) = y^{1-\alpha}(t) = y^{-4}(t) \implies u'(t) = -4y'(t)y^{-5}(t)$.

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Rewrite the ODE as:

$$\frac{y'}{y^5} = \frac{y}{y^5} + 2\frac{\cancel{y^5}}{\cancel{y^5}} \implies \frac{-u'}{4} = u+2 \implies u' = -4(u+2) \rightarrow 1^{\text{st}} \text{ order, linear ODE}$$

Solving now in u returns: $u(t) = Ce^{-4t} - 2$, $C \in \mathbb{R}$. The solution of the original ODE is thus:

$$y(t) = \pm u^{-1/4}(t) = \pm \frac{1}{(Ce^{-4t} - 2)^{1/4}}$$

Riccati equation[*]

A first order (non-linear) equation of the form

$$y'(t) = a(t)y(t) + b(t)y(t)^2 + c(t), \text{ with } a, b, c \in \mathrm{C}(I) \tag{2}$$

is called **Riccati differential equation**.

Suppose we already have a **particular solution y_p** of (2). Then the difference $v(t) := y(t) - y_p(t)$ solves the Bernoulli ODE $v'(t) - v(t)[a(t) + 2b(t)y_p(t)] = b(t)v(t)^2$.

Thus setting $\boldsymbol{u(t) := v^{-1}(t) = \frac{1}{y(t)-y_p(t)}}$, we find the first order linear ODE

$$u'(t) = -u(t)[a(t) + 2b(t)y_p(t)] - b(t) \tag{3}$$

to be solved in u.

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[*] Studied by the Venetian mathematician Jacopo Riccati (1676-1754)

Example 2

Find the general solution of the ODE $y' = -y^2 + \frac{2}{t^2}$, for $y = y(t)$ and $t > 0$.

It is a Riccati equation with $a(t) = 0$, $b(t) = -1$ and $c(t) = \frac{2}{t^2}$. Taking $y_p(t) = \frac{k}{t}$, $k \in \mathbb{R}$ as Ansatz for a particular solution, find the appropriate k.

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By substitution we see that $y_p := -1/t$ is a solution. Let $u := \frac{1}{y - y_p}$, from which equation (3) becomes:

$$u'(t) = -u(t)[0 + 2(-1)(-1/t)] + 1 = 1 - 2u(t)/t$$

Solving the latter in u yields: $u(t) = \frac{t^3 + C}{3t^2}$, for $C \in \mathbb{R}$.

Returning to y we obtain: $y(t) = y_p(t) + \frac{1}{u(t)} = -\frac{1}{t} + \frac{3t^2}{t^3 + C} = \frac{2t^3 - C}{t(t^3 + C)}$,
plus $y(t) = y_p(t) = -1/t$.

Exact differential equations

Let $D \subseteq \mathbb{R}^2$ open. A first order ODE of the form

$$f(t, y(t)) + g(t, y(t))y'(t) = 0 \tag{4}$$

is called **exact** in D if there exists a C^1 function $\psi : D \to \mathbb{R}$ such that

$$\begin{cases} \frac{\partial\psi}{\partial t}(t, y) = f(t, y) \\ \frac{\partial\psi}{\partial y}(t, y) = g(t, y), \end{cases}$$

for all $(t, y) \in D$. In such case, a C^1 function y s.t. $(t, y(t)) \in D$ $\forall t$ solves (4) if and only if

$$\frac{\mathrm{d}\psi}{\mathrm{d}t}(t, y(t)) = \frac{\partial\psi}{\partial t}(t, y(t)) + \frac{\partial\psi}{\partial y}(t, y)\frac{\mathrm{d}y}{\mathrm{d}t} = f(t, y) + g(t, y)y'(t) = 0$$

$$\iff \psi(t, y(t)) = C, \qquad \text{for some } C \in \mathbb{R}.$$

If (4) is exact, the function ψ is called **potential** of the ODE.

Necessary and sufficient conditions to exact ODEs

Determining if an ODE of the kind

$$f(t, y(t)) + g(t, y(t))y'(t) = 0 \tag{4}$$

is exact by applying the definition may not be immediate.
For this reason, we make use of the following criterion:

Theorem (integrability criterion for exact ODEs)

If f and g are $\mathrm{C}^1(D)$ with $D \subseteq \mathbb{R}^2$ simply connected, then:

$$(4) \textit{ is exact in } D \iff \frac{\partial f}{\partial y}(t, y) = \frac{\partial g}{\partial t}(t, y), \quad \textit{for all } (t, y) \in D.$$

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Example: the differential equation $\mathbf{2ty(t)} + (\boldsymbol{t}^2 + \boldsymbol{y}(\boldsymbol{t})^2 + 3\boldsymbol{y}(\boldsymbol{t}))y'(t) = 0$ is exact in $\mathbb{R}^2$, since $\frac{\partial \mathbf{f}}{\partial y}(t, y) = 2t = \frac{\partial \mathbf{g}}{\partial t}(t, y)$ for every $(t, y) \in \mathbb{R}^2$.

Equations without explicit dependent variable

Consider an ODE of order $m > 1$ in which the dependent variable does NOT appear, namely (in the explicit form)

$$y^{(m)} = f(t, y', y'', \dots, y^{(m-1)}).$$

Letting $\boldsymbol{u(t) := y'(t)}$, we reduce the order of the equation by one.

Specifically, for $m = 2$ it is $y'' = f(t, y')$ and applying the substitution we find

$$u' = f(t, u) \to \textbf{first order} \text{ ODE in } u = u(t)$$

to be solved with respect to u (by the formula, or whenever possible by separation of variables). Finally bring back to y.

Example 3

In order to find the general solution of the ODE

$$y'' + 2(y')^2 = 0 \rightarrow \text{ (non-linear) } \textbf{second order} \text{ ODE, no explicit } y$$

we substitute $u(t) := y'(t)$ and find

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Notice that $u \equiv 0$ is a solution (corresponding to $y \equiv C$). Suppose then $u \neq 0$ and apply separation of variables to obtain:

$$\int \frac{-1}{2u^2}\,\mathrm{d}u = \int \frac{u'(t)}{2u(t)^2}\,\mathrm{d}t = \int \mathrm{d}t \iff$$

$$\iff \frac{1}{2u} = t + C_1 \iff u(t) = \frac{1}{2t + C_1} = y'(t) \implies$$

$$y(t) = \int \frac{1}{2t + C_1}\,\mathrm{d}t = \frac{1}{2}\ln|2t + C_1| + C_2 \text{ OR } y(t) = C \text{ is the gen. sol.}$$

Exercise 1

For any of the following differential equations:

- identify if it is a Bernoulli or a Riccati equation;
- determine the appropriate substitution to get a linear ODE;
- solve the new equation and thus the original one.

(i) $y' + ty - ty^3 = 0;$

(ii) $t^2u' - u^4 = tu,\ t > 2;$

(iii) $y' + 6y^2 = 1/t^2,\ t > 0;$ Hint: look for sol. of the kind $y_p(t) := \dfrac{\alpha}{t} + \beta;$

(iv) $x' - e^t\sqrt{x} = -2x,\ x > 0;$

(v) $x^3u' + x^2u - u^2 = 2x^4,\ x > 1.$ Hint: look for a 2nd degree polynomial sol.

Exercise 2

For any of the following differential equations, determine:

- if they are exact or not;
- for each exact equation compute a corresponding potential;
- whenever possible, determine the solution of the exact ODEs by solving the (algebraic) level set equation for the potential.

$$
\begin{aligned}
&(i)\ 2tu + (t^2+3)u' = 0;\\
&(ii)\ \cos(t)y' + y - t^3 y = 0;\\
&(iii)\ y + (x-1)y' = -2x,\ x > 1;\\
&(iv)\ 3x^2 + y^2 + 2y(1+x)y' = 0,\ x > 3;\\
&(v)\ -y\cos(t) = y'(\sin(t) + \sin(y) + y\cos(y));\\
&(vi)\ 2 - (3x^2 + u - u^2)u' + x^3 = 3xu^2.
\end{aligned}
$$

Exercise 3

Solve the following initial value problems of second order differential equations.

$$\text{(i)}\begin{cases} y''(t) - 8y'(t) = 8; \\ \text{y(0)} = 1; \\ y'(0) = 3. \end{cases}$$

$$\text{(ii)}\begin{cases} \cos(t)u''(t) + \sin(t)u'(t) = 3\cos^2(t), \;\; t \in (0, \pi/2); \\ \text{u(0)} = 2; \\ u'(0) = -1. \end{cases}$$

EXERCISE 1

Bernoulli ODE

$$(i) \quad \left[y' + ty - ty^{\overset{\alpha}{3}} = 0 \right] \ 1^{st} \text{ order} \qquad \alpha = 3$$

$$y' = \underbrace{-t}_{a(t)} y + \underbrace{t}_{b(t)} y^3$$

Set $u(t) := y^{1-\alpha}(t) = y^{-2}(t) = \frac{1}{y^2(t)}$

$$u' = -2 \cdot y^{-3} \cdot y' = \frac{-2y'}{y^3} \quad \downarrow \quad -\frac{1}{2}u' = \frac{y'}{y^3}$$

Divide the terms in the ODE by y^3:

$$\frac{y'}{y^3} = \frac{-t \cdot y}{y^3} + \frac{t\cancel{y^3}}{\cancel{y^3}} = \frac{-t}{y^2} + t$$

$$\downarrow \quad -\frac{u'}{2} = -tu + t \Rightarrow \left[u' = \underbrace{2t}\,u \ \underbrace{-2t} \right] \ (*) \quad 1^{st} \text{ order ODE linear}$$

$u' = 2t(u-1)$ ~ separable variable ODE

- If $u \equiv 1$ ~ $u' = 0 = 2t(1-1) \Rightarrow u \equiv 1$ solves the ODE
- Suppose $u \neq 1$: $\frac{u'}{u-1} = 2t$ ~ $\int \frac{du}{u-1} = \int \frac{u'dt}{u-1} = \int 2t\,dt$

$$\ln|u-1| = t^2 + c \Rightarrow |u-1| = e^{t^2+c} = e^c \cdot e^{t^2}, \quad c \in \mathbb{R} \rightarrow e^c > 0$$

$$u - 1 = \boxed{\pm e^c}\, e^{t^2} \qquad c_1 := \pm e^c \qquad \pm e^c \neq 0$$

$$u = 1 + c_1 e^{t^2} \qquad c_1 \neq 0$$ ~ notice that for $c_1 = 0$, $u \equiv 1$!

Altogether, we get: $\left[u(t) = 1 + c \cdot e^{t^2}, \quad c \in \mathbb{R} \right]$ gen. sol. of (*)

Recall: $u(t) = \frac{1}{y^2(t)}$ ~ $\left[y(t) = \pm \frac{1}{\sqrt{u(t)}} = \pm \frac{1}{\sqrt{1 + ce^{t^2}}}, \quad c \in \mathbb{R} \right]$

gen. sol. of the original ODE

(iii) $y' + 6y^2 = \frac{1}{t^2}$, $t>0$ ← Riccati ODE

$a(t)=0$

$$y' = \underset{b(t)}{\underset{\parallel}{\boxed{-6}}}\, y^2 \underset{c(t)}{\underset{\parallel}{\boxed{+\frac{1}{t^2}}}}$$

- y_p particular sol. of the ODE ⇝ $y_p(t) = \frac{\alpha}{t} + \beta$, α, β to be determined

$y_p' = -\frac{\alpha}{t^2}$

Substitute:

$$-\frac{\alpha}{t^2} + 6\left(\frac{\alpha}{t}+\beta\right)^2 = \frac{1}{t^2}, \ \forall t \Rightarrow \boxed{y_p(t) = \frac{1}{2t}} \left(\vee \quad y_p(t) = -\frac{1}{3t}\right)$$

↑ by comparison of the coefficients

. . . .

$\begin{cases} \alpha = \dots \\ \beta = \dots \end{cases}$

↖ for example, choose this as y_p
(the choice is arbitrary, with $y_p = -\frac{1}{3t}$ we will get to the same gen. sol. of the ODE!)

Let $u(t) := \frac{1}{y(t) - y_p(t)} = \frac{1}{y(t) - \frac{1}{2t}}$

Apply the formula to obtain:
↳ or compute, if you want!!

$$u'(t) = -u(t)\left[\overset{0}{\overset{\parallel}{\cancel{a}}}(t) + 2b(t)\cdot y_p(t)\right] - b(t)$$

$$u' = +u\left[\cancel{2}\cdot(+6)\cdot\frac{1}{\cancel{2}t}\right] + 6 = \frac{6}{t}u + 6$$

(**)

$$\left[u'(t) = \frac{6}{t}u(t) + 6\right]$$ 1st order linear ODE in u

Solution:

$u(t) = e^{A(t)}\left[B^*(t) + C\right]$ ← formula for 1st order lin. ODES (inhomogeneous)

$$A(t) = \int \frac{6}{t}dt = 6\ln(t) \overset{0}{\overset{\parallel}{+\cancel{c}}} \quad \Big| \quad B^*(t) = \int 6 e^{-6\ln(t)} dt = 6\int e^{-\ln(t^6)}dt = 6\int \frac{1}{e^{\ln(t^6)}}dt =$$

$$= 6\int t^{-6}dt = 6t^{-5}\left(-\frac{1}{5}\right) = -\frac{6}{5t^5} + \underset{0}{\underset{\parallel}{\cancel{c}}}$$

$$\left[u(t) = e^{\ln(t^6)}\left(\frac{-6}{5t^5} + C\right) = Ct^6 - \frac{6}{5}t, \ C\in\mathbb{R}\right]$$ → sol. of (**)

Substitute: $u = \frac{1}{y - y_p} \Rightarrow \left[\begin{array}{l} y = \frac{1}{u} + y_p = \frac{1}{Ct^6 - \frac{6}{5}t} + \frac{1}{2t} \\ \quad\vee \\ y = \frac{1}{2t} \end{array}\right]$ → gen. sol. of (iii)

EXERCISE 2

(i) $\overset{=f}{(2t\cdot u)}+\overset{=g}{(t^2+3)}u'=0 \rightsquigarrow$ is it exact in $D=\mathbb{R}^2$?

$$\left(u' = \frac{-2t}{t^2+3}\,u\right) \longrightarrow \begin{array}{l} f(t,u)=2t\cdot u \\ g(t,u)=t^2+3 \end{array} \qquad f,g\in C^\infty(\mathbb{R}^2)$$

$$\frac{\partial f}{\partial u}=2t \;\underset{\text{"}}{,}\;\forall t \Rightarrow \text{by criterion, the ODE is exact!}$$

$$\frac{\partial g}{\partial t}=2t$$

We look now for a potential of the ODE: a function $\Psi=\Psi(t,u)$, $\Psi:\mathbb{R}^2\to\mathbb{R}$ such that

$$\begin{cases} \dfrac{\partial\Psi}{\partial t}=f \\ \dfrac{\partial\Psi}{\partial u}=g \end{cases} \qquad \forall(t,u)\in\mathbb{R}^2$$

Solve:

$$f(t,u)=2tu \underset{\text{set}}{\overset{\uparrow}{=}} \frac{\partial\Psi}{\partial t}(t,u) \Rightarrow \Psi(t,u)=\int 2tu\,\boxed{dt} = t^2u + c(u) = t^2u+3u+\cancel{c}^{\,0}$$

potential

$$g(t,u)=t^2+3 \overset{\downarrow}{=} \frac{\partial\Psi}{\partial u}(t,u) = t^2+c'(u) \Rightarrow \cancel{t^2}+c'(u)=\cancel{t^2}+3$$

$$c(u)=3u+\cancel{c}^{\,0}, \quad c\in\mathbb{R}$$

without loss of generality, we may take $c=0 \rightsquigarrow$ we are fixing ⓐ potential Ψ

We solve the ODE employing the property of the potential Ψ:

$$\frac{d\Psi(t,u(t))}{dt}=0 \rightsquigarrow \boxed{\Psi(t,u(t))=K}, \quad \text{for } u=u(t) \text{ sol. of the exact ODE}, \; K\in\mathbb{R} \text{ constant}$$

$$\hookrightarrow t^2\cdot u+3u=K$$

$$u(t^2+3)=K \Rightarrow \left[u(t)=\frac{K}{t^2+3}\right] \text{ is the gen sol. of the ODE!}$$